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Aerodynamic sound generation by turbulent flow in the vicinity of a scattering half plane

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The presence of the edge of a half plane in a turbulent fluid results in a large increase in the noise generated by that fluid at low Mach numbers. The parameter which is important is the product $2k\bar{r}_0$, where $\bar{r}_0$ is the distance of the centre of an eddy from the edge. Eddies which satisfy the inequality $2k\bar{r}_0 \ll 1$ have the sound output of the quadrupoles associated with the fluid motion in a plane normal to the edge increased by the factor $(k\bar{r}_0)^{-3}$. There is no enhancement of the sound from the longitudinal quadrupoles with axes parallel to the edge: the rr, θr and $\theta\theta$ quadrupoles are the dominant sound sources. The far field sound intensity induced by these sources depends upon the fifth power of a typical fluid velocity. The intensity has a directional dependence on $\cos^2 \frac{1}{2}\theta$ if the half plane is rigid and $\sin^2 \frac{1}{2}\theta$ if it is a pressure release surface, $\theta = 0$ being a direction in the half plane.

If the eddies are far from the edge so that $(k\bar{r}_0)^{\frac{1}{2}} \gg 1$ then the farfield sound has the same features as would be predicted by geometrical acoustics. The edge does not produce any significant sound amplification.

1. Introduction

Lighthill (1952, 1954), in his theory of aerodynamic sound, modelled the problem of sound generation by turbulence in an exact analogy with sound radiated by a volume distribution of acoustic quadrupoles embedded in an ideal acoustic medium. The strength density of the equivalent quadrupoles is Lighthill's stress tensor which is essentially the unsteady component of the Reynolds stress in low Mach number flows. Curle (1955) showed how the presence of boundary surfaces could be accounted for by additional surface distributions of dipole and monopole sources. A dimensional analysis based on the idea that the only velocity and length scales in the problem are set by those in the turbulence yields the well-known laws that the intensity of sound generated by free turbulence increases with the eighth power of flow velocity, while that induced by unsteady surface forces increases in proportion to the sixth power of flow velocity. At sufficiently low speeds, the sound induced by any surface in the flow is therefore dominant, and the problem is one of considerable practical importance in both aeronautical and marine applications. On the general grounds that the Reynolds number in most practical flows is very large the direct effects of viscosity are considered to be of minor importance, so that the aerodynamic

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noise problem is posed as the question of estimating the radiation field of known quadrupole sources near any surfaces that may be present according to inviscid propagation equations.

An important class of problems that have so far not been included in the general theory is characterized by local regions where the relevant length and velocity scales are *not* set by those in the turbulence. Such a case is found whenever the body has sharp edges, the edges acting as scattering centres in the vicinity of which the field is governed by diffraction effects according to linearized propagation equations. When this is so, the premise of Curle's dimensional analysis is violated and the issue stands as an open question. The treatment of this general class requires rather a different technique from that used previously and must take into account the details of the potential field in the vicinity of the scattering zone. A problem in this class is considered in this paper and indicates an interesting result that the field associated with edge scattering is more powerful than both the direct Lighthill field and the surface dipole field of the usual Curle type. This conclusion essentially rests on the potential field singularity of the diffraction problem at the edge and would be substantially modified if any type of 'Kutta' condition were invoked to limit its effect. At low enough frequencies it may well be that the edge flow is determined by viscous effects—which are not considered in this paper, but there can be little doubt that at sufficiently high frequencies the problem is more properly posed as one of the usual diffraction type. This is what we do in considering the problem of what potential field is radiated by a quadrupole distribution in the vicinity of a sharp edged thin half plane. We solve the problem with the aid of the known exact Green's function and conclude that the edge scattered field is proportional in intensity to the fifth power of flow velocity. As such it is likely to be the dominant sound source at sufficiently low flow speeds.

2. The solution of Lighthill's equation

The basic equation which describes aerodynamic noise generation and propagation and which is taken as the starting point of this analysis is due to Lighthill (1952)

$$\nabla^2\rho-\frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2}=-\frac{1}{c^2}\frac{\partial^2}{\partial y_i\,\partial y_j}(\rho v_i v_j+p_{ij}-c^2\rho\delta_{ij}), \tag{1}$$

where ρ is the fluid density, (v_1, v_2, v_3) the velocity vector, c the sound speed in the undisturbed fluid and p_{ij} the compressive stress tensor. We shall initially assume that viscous effects are negligible so that we set p_{ij} equal to $p\delta_{ij}$, where p is the isotropic pressure in the fluid. If we further suppose that changes in p are exactly balanced by changes in $c^2\rho$ then Lighthill's equation can be written

$$\nabla^2 p-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=-\frac{\partial^2\rho v_i v_j}{\partial y_i\,\partial y_j}. \tag{2}$$

We seek a solution of this equation when there is a rigid, vanishingly thin, half-plane immersed in an otherwise unbounded fluid.

If we define the generalized Fourier transform of the function $f(t)$ as

$$f^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt,$$

then Lighthill's equation can be written as the inhomogeneous Helmholtz equation

$$\nabla^2 p^* + k^2 p^* = -\left[\frac{\partial^2 \rho v_i v_j}{\partial y_i \partial y_j}\right]^*, \tag{3}$$

where $k = \omega/c$.

The presence of the rigid half plane gives the boundary condition that the normal velocity vanishes at the surface. The solution of (3) with this boundary condition can be written down at once in terms of a Green's function, G, whose normal derivative vanishes on the half-plane. It is:

$$p^*(\mathbf{x}, \omega) = \frac{1}{4\pi}\int \left(\frac{\partial^2 \rho v_i v_j}{\partial y_i \partial y_j}\right)^* G\, dV(\mathbf{y}) + \frac{1}{4\pi}\int \frac{\partial p^*}{\partial n} G\, dS, \tag{4}$$

$$\left.\begin{aligned} &\text{where} && (\nabla^2 + k^2)\, G = -4\pi\delta(\mathbf{x}-\mathbf{y}),\\ &\text{with} && \frac{\partial G}{\partial n} = 0 \quad \text{on the half-plane.}\end{aligned}\right\} \tag{5}$$

The volume integral in (4) is strictly over all space, but as $(\partial^2 \rho v_i v_j/\partial y_i\, \partial y_j)^*$ is considered non-zero only within the turbulence, the volume integral need be evaluated only over that region. If we now complete the divergences in (4), and convert the volume divergence integrals into surface integrals by the use of Gauss's theorem we find that the surface integrals vanish because of the condition that there is no normal velocity on the half-plane and we are left with

$$4\pi p^*(\mathbf{x}, \omega) = \int (\rho v_i v_j)^* \frac{\partial^2 G}{\partial y_i \partial y_j}\, dV(\mathbf{y}). \tag{6}$$

In cylindrical polars this is

$$\begin{aligned} 4\pi p^*(r,\theta,z,\omega) = \int \Bigg\{ & \rho v_r^2 \frac{\partial^2 G}{\partial r_0^2} + \rho v_z^2 \frac{\partial^2 G}{\partial z_0^2} + \rho v_r v_z \left[\frac{\partial}{\partial r_0}\left(\frac{\partial G}{\partial z_0}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial G}{\partial r_0}\right)\right] \\ & + \rho v_r v_\theta \left[\frac{\partial}{\partial r_0}\left(\frac{1}{r_0}\frac{\partial G}{\partial \theta_0}\right) + \frac{2}{r_0}\frac{\partial}{\partial \theta_0}\left(\frac{\partial G}{\partial r_0}\right) - \frac{1}{r_0^2}\frac{\partial G}{\partial \theta_0}\right] \\ & + \rho v_\theta v_z \left[\frac{1}{r_0}\frac{\partial}{\partial \theta_0}\left(\frac{\partial G}{\partial z_0}\right) + \frac{\partial}{\partial z_0}\left(\frac{1}{r_0}\frac{\partial G}{\partial \theta_0}\right)\right] \\ & + \rho v_\theta^2 \left(\frac{1}{r_0^2}\frac{\partial^2 G}{\partial \theta_0^2} + \frac{1}{r_0}\frac{\partial G}{\partial r_0}\right)\Bigg\}^* dV_0, \end{aligned} \tag{7}$$

where $dV_0 = r_0\, dr_0\, d\theta_0\, dz_0$.

The particular cylindrical co-ordinate system which we use is illustrated in figure 1.

We restrict our attention to field points which are many wavelengths both from the turbulent region and from the edge of the half plane. That is, we suppose $kr \gg 1$ and $r \gg r_0$.

Macdonald (1915) has shown that the solution of (5) takes, in the far field, the form:

$$G = \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\left\{\frac{e^{-ikR}}{R}\int_{-\infty}^{u_R} e^{-iu^2}\,du + \frac{e^{-ikR'}}{R'}\int_{-\infty}^{u_{R'}} e^{-iu^2}\,du\right\}, \tag{8}$$

where $$U_R = 2\left(\frac{krr_0}{D+R}\right)^{\frac{1}{2}}\cos\frac{\theta-\theta_0}{2} = \pm\,[k(D-R)]^{\frac{1}{2}}$$

and $$u_{R'} = 2\left(\frac{krr_0}{D+R'}\right)^{\frac{1}{2}}\cos\frac{\theta+\theta_0}{2} = \pm\,[k(D-R')]^{\frac{1}{2}}.$$

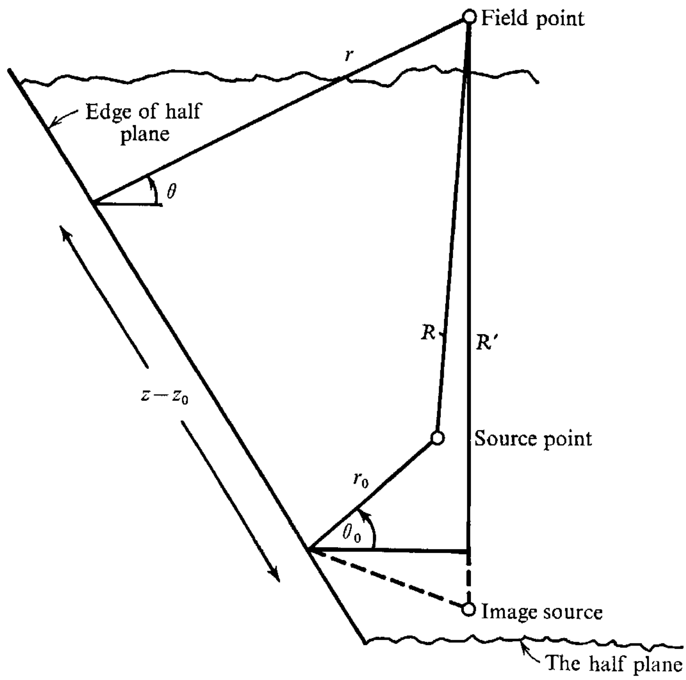

FIGURE 1. The co-ordinate system.

R is the separation of the source point (r_0, θ_0, z_0) and the field point (r, θ, z), i.e.

$$R = \{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0) + (z-z_0)^2\}^{\frac{1}{2}}.$$

R' is the separation of the specular image source point in the plane containing the half plane, $(r_0, -\theta_0, z_0)$, and the field point:

$$R' = \{r^2 + r_0^2 - 2rr_0\cos(\theta+\theta_0) + (z-z_0)^2\}^{\frac{1}{2}},$$

$$D = \{(r+r_0)^2 + (z-z_0)^2\}^{\frac{1}{2}}.$$

It can be shown that D is the shortest distance between the source and field points travelling *via* the edge.

G is similar to the Green's function for an infinite rigid plane with the difference that now each term is weighted by a Fresnel integral whose magnitude can vary between 0 and 1 (approximately). Any enhancement of the sound field from that produced by turbulence in free space or near a rigid plane can only arise from the derivatives of these integrals, and in particular, from the derivatives of u_R and $u_{R'}$.

The limits of integration u_R and $u_{R'}$ may be simplified by noting that any derivative of the factors $(D+R)^{-\frac{1}{2}}$ or $(D+R')^{-\frac{1}{2}}$ does not appear in the farfield representation of p^* when G is substituted into (7). We may therefore use at once the farfield approximations:

$$D+R \approx 2\{r^2+(z-z_0)^2\}^{\frac{1}{2}}$$

and

$$D+R' \approx 2\{r^2+(z-z_0)^2\}^{\frac{1}{2}},$$

to write

$$u_R = (2kr_0 \sin\phi)^{\frac{1}{2}} \cos\tfrac{1}{2}(\theta-\theta_0),$$

$$u_{R'} = (2kr_0 \sin\phi)^{\frac{1}{2}} \cos\tfrac{1}{2}(\theta+\theta_0),$$

where

$$\sin\phi = \frac{r}{\sqrt{[r^2+(z-z_0)^2]}}.$$

The independence in the farfield of u_R and $u_{R'}$ on the co-ordinate z_0 excludes the possibility that the term

$$\int (\rho v_z^2)^* \left(\frac{\partial^2 G}{\partial z_0^2}\right) dV_0$$

in (7) might have a magnitude much greater than if G were just that of a rigid plane. Accordingly, we can conclude immediately that the edge does not result in any significant enhancement of the sound produced by longitudinal quadrupoles aligned parallel with the edge.

Also, another deduction can be made from the general form of G, namely, that the sound field at points on the plane $\theta = \pi$ has exactly the same features as sound from free turbulence. This may be seen as follows. On the plane $\theta = \pi$ (which is the half plane complementary to the material half plane) the field point is equidistant from the real and image source points. That is, $R = R'$ and so,

$$G = \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}} \frac{e^{-ikR}}{R}\left\{\int_{-\infty}^{2\left(\frac{krr_0}{D+R}\right)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0} e^{-iu^2}du + \int_{-\infty}^{-2\left(\frac{krr_0}{D+R}\right)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0} e^{-iu^2}du\right\}$$

$$= \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}} \frac{e^{-ikR}}{R}\left\{2\int_0^{\infty} e^{-iu^2}du + \int_0^{2\left(\frac{krr_0}{D+R}\right)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0} e^{-iu^2}du + \int_0^{-2\left(\frac{krr_0}{D+R}\right)^{\frac{1}{2}}\sin\frac{1}{2}\theta_0} e^{-iu^2}du\right\}$$

$$= \frac{e^{-ikR}}{R}, \quad \text{the Green's function for an unbounded fluid.}$$

We are now ready to substitute the expression for G given by (8) into the equation for $p^*(r,\theta,z,\omega)$. We shall consider separately the two cases of turbulence well within a typical acoustic wavelength of the edge and many wavelengths away from the edge.

3. The noise from eddies very near the edge

We first consider the case of eddies (that is, regions of the turbulence over which fluctuations of velocity are highly correlated) which are well within a wavelength of the edge. By this we mean that every part of the eddy satisfies the inequality

$$2kr_0 \ll 1.$$

The Fresnel integral $$\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\int_{-\infty}^{X} e^{-iu^2}\,du$$
has the series expansion

$$\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\int_{-\infty}^{X} e^{-iu^2}\,du = \frac{1}{2}+\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\,X(1-\tfrac{1}{3}(i)X^2+O(X^4))$$

(see, for example, the introduction to Pearcey's (1956) tables).

Hence we can write,

$$G = \frac{e^{-ikR}}{R}\left\{\frac{1}{2}+\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}(2kr_0\sin\phi)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\theta_0)\,(1+O(kr_0))\right\}$$
$$+\frac{e^{-ikR'}}{R'}\left\{\frac{1}{2}+\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}(2kr_0\sin\phi)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta+\theta)_0\,(1+O(kr_0))\right\},$$

or, noting that

$$kR' = kR+2kr_0\sin\theta_0\sin\theta+O(kr_0^2/R),$$

we can replace R' by R:

$$G = \frac{e^{-ikR}}{R}\left\{1+\frac{2e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}(2kr_0\sin\phi)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta_0\cos\tfrac{1}{2}\theta+O(kr_0)\right\}. \qquad (9)$$

When this expression for G is fed into equation (7) we obtain terms containing $(2kr_0)^{-\frac{3}{2}}$, $(2kr_0)^{-\frac{1}{2}}$ or positive powers of $2kr_0$. Under the condition $2kr_0 \ll 1$ the dominant terms are those containing $(2kr_0)^{-\frac{3}{2}}$ and it is these terms we retain when we write

$$-4\pi p^*(r,\theta,z;\omega) = k^2\frac{2e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}(\sin\phi)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta$$
$$\times\int\{\rho v_r^2\cos\tfrac{1}{2}\theta_0-\rho v_\theta^2\cos\tfrac{1}{2}\theta_0-2\rho v_r v_\theta\sin\tfrac{1}{2}\theta_0\}^*\,(2kr_0)^{-\frac{3}{2}}\frac{e^{-ikR}}{R}\,dV_0, \quad (10)$$

where the volume integral is evaluated over those eddies which satisfy $2kr_0 \ll 1$.

This equation is the basic result of this section. It is exactly the same result as is obtained by performing the differentiations on the Green's function in the form (8) and then picking out the dominant terms under the condition $2kr_0 \ll 1$. It should be compared with the corresponding equation which is applicable to an unbounded turbulent fluid:

$$4\pi p^*(r,\theta,z;\omega) = -k^2\int\{\rho v_r^2\cos^2(\theta-\theta_0)+\rho v_\theta^2\sin^2(\theta-\theta_0)$$
$$+2\rho v_r v_\theta\cos(\theta-\theta_0)\sin(\theta-\theta_0)$$
$$+\text{similar terms involving } \textit{each} \text{ of the remaining}$$
$$\text{Reynolds' stresses}\}^*\,\frac{e^{-ikR}}{R}\,dV_0. \qquad (11)$$

The first point to notice is that the integrand of (10) contains the large factor $(2kr_0)^{-\frac{3}{2}}$. This has the consequence that the farfield acoustic pressure levels when there is an edge in the turbulent region may be considerably greater than when there is none. 'May be', because the pressure field has a different directionality to that of the radiation field of an eddy in free turbulence.

A further point to notice is that the different Reynolds stresses are differently affected by the half plane. The stresses ρv_r^2, ρv_θ^2 and $\rho v_r v_\theta$ produce pressure fields which are greater by a factor of order $(2kr_0)^{-\frac{3}{2}}$ than the free turbulence values; the stresses $\rho v_r v_z$ and $\rho v_\theta v_z$ (which are not shown in (10)) are increased by the smaller factor $(2kr_0)^{-\frac{1}{2}}$, while the stress ρv_z^2 has just the pressure field we would expect if the half plane was an infinite plane (i.e. had no edge).

Following Lighthill, we regard the turbulence as divided into regions within which each of the products $(\rho v_r^2)^*$, $(\rho v_\theta^2)^*$ and $(\rho v_r v_\theta)^*$ is perfectly correlated, the size of each region being very much less than an acoustic wavelength. For the sound pressure from such a region we can write

$$4\pi p^*(r,\theta,z;\,\omega) = k^2\frac{2e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}(\sin\phi)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta\frac{e^{-ikR}}{R}$$
$$\times\left[(\rho v_r^2-\rho v_\theta^2)^*\int\cos\tfrac{1}{2}\theta_0(2kr_0)^{-\frac{3}{2}}\,dV_0-2(\rho v_r v_\theta)^*\int\sin\tfrac{1}{2}\theta_0(2kr_0)^{-\frac{3}{2}}\,dV_0\right], \quad (12)$$

where the volume integrals are now to be evaluated over the region of perfect correlation. If such a region is supposed to occupy the space $r_1 < r_0 < r_2$, $\theta_1 < \theta_0 < \theta_2$ and $z_1 < z_0 < z_2$ then a good approximation to these volume integrals is

$$2^{\frac{1}{2}}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\frac{\beta}{2}(k\bar{r}_0)^{-\frac{3}{2}}\,V,$$

where the cos or sin is to be taken if the original integral contained a cos or a sin, V is the volume of the eddy and $\beta = \frac{1}{2}(\theta_1+\theta_2)$ and $\bar{r}_0 = \frac{1}{2}(r_1+r_2)$. $\bar{r}_0$ and β may be regarded as the r_0, θ_0 co-ordinates of the centre of the eddy.

The volume integrals may also be evaluated if it is supposed that the eddy is a cylinder centred on the edge of the half plane. The integrals containing $\cos\frac{1}{2}\theta_0$ vanish but

$$\int\sin\tfrac{1}{2}\theta_0(2kr_0)^{-\frac{3}{2}}\,dV_0 = \frac{2^{\frac{3}{2}}}{\pi}(k\delta)^{-\frac{3}{2}}\,V$$
$$= 2^{\frac{1}{2}}(1\cdot 3k\delta)^{-\frac{3}{2}}\,V,$$

where 2δ is the diameter of the cylinder. This result indicates that a lower bound for $\bar{r}_0$ should be $1\cdot 3\delta$, or, roughly, δ.

If we write $v_r = U_r+u_r$, $v_\theta = U_\theta+u_\theta$ and $v_z = U_z+u_z$ where the flow near the edge is regarded as being composed of a steady, time independent part (U_r, U_θ, U_z) and a fluctuating part (u_r, u_θ, u_z) then, for example,

$$(\rho v_r^2)^* = \rho_0(U_r^2+2U_r u_r+u_r^2)^*$$
$$\approx 2\rho_0 U_r u_r^*,$$

where ρ has been set equal to ρ_0, the density of the undisturbed fluid. U_r^2, because it is independent of time, makes no contribution and the term, u_r^2 is neglected because it is smaller than the term $U_r u_r$ by the factor of α, the normalized turbulence intensity. That is u_r is of order α times a typical flow velocity (U say).

Instead of (12) we may now write the approximate relation

$$4\pi p^* = 4\left(\frac{2}{\pi}\right)^{\frac{1}{2}} k^2 \cos\tfrac{1}{2}\theta_0 (\sin\phi)^{\frac{1}{2}} \frac{e^{-ikR}}{R} \rho_0 U^2 \alpha \sin\bar{\theta} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} \frac{\beta}{2} (k\bar{r}_0)^{-\frac{3}{2}} V, \tag{13}$$

where $\bar{\theta}$ is the angle the mean flow makes with the edge of the half plane. From this we can obtain an approximate formula for the farfield acoustic intensity that neglects any effects of cross-correlation between individual terms of (12). It is

$$I(r,\theta,z;\omega) = \frac{k^4 \sin\phi \cos^2(\frac{1}{2}\theta)\rho_0 U^4 \alpha^2 \sin^2\bar{\theta} \begin{Bmatrix}\cos^2\\ \sin^2\end{Bmatrix} \frac{\beta}{2} V^2}{\pi^3 c R^2 (k\bar{r}_0)^3}. \tag{14}$$

Setting $\bar{r}_0$ equal to the correlation radius we find I has a maximum value of

$$I_{\max} \approx \frac{\rho k U^4 V^2 \alpha^2}{\pi^3 c R^2 \delta^3}. \tag{15}$$

The typical frequency of the turbulent source is of order $U/2\delta$ so that k is of order $\pi U/c\delta$. *Thus the scattered intensity increases in proportion to the fifth power of the fluid velocity*, U. This is a new result which should be compared to the eighth power law obtained from free turbulence, or turbulence supported by an infinite plane, and the sixth power law obtained from the usual estimations of Curle's (1955) surface dipole term for acoustically compact surfaces. Curle's solution of Lighthill's equation in the presence of surfaces is undoubtedly correct. However, his result is difficult to interpret for non-compact surfaces because it is not possible to estimate quantitatively the 'dipole' term for surfaces which are not small compared with an acoustic wavelength. Dimensional arguments are altogether too crude: for infinite planes they overestimate the sound and for sharp edged semi-infinite planes they underestimate it.

4. The noise from eddies remote from the edge

We now consider the effect of the half plane on the noise from those eddies which are far enough from the edge for the inequality $(k\bar{r}_0)^{\frac{1}{2}} \gg 1$ to hold. $\bar{r}_0$ is again the distance of the centre of the eddy from the edge.

We write G in Macdonald's form:

$$G = \frac{e^{-ikR}}{R} I_R + \frac{e^{-ikR'}}{R'} I_{R'},$$

where

$$I_R = \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}} \int_{-\infty}^{u_R} e^{-iu^2}\,du \tag{16}$$

and

$$I_{R'} = \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}} \int_{-\infty}^{u_{R'}} e^{-iu^2}\,du.$$

When G is substituted into (7) for p^* and the factor k^2 abstracted from each term, three distinct sets of terms are obtained; those containing the factor I_R or $I_{R'}$, those containing the factor $(kr_0)^{-\frac{1}{2}}$ and, finally, those with the factor $(kr_0)^{-\frac{3}{2}}$.

For example,

$$\int (\rho v_\theta^2)^* \left\{\frac{1}{r_0^2}\frac{\partial^2}{\partial\theta_0^2}+\frac{1}{r_0}\frac{\partial}{\partial r_0}\right\}\frac{e^{-ikR}}{R}I_R\,dV_0 = \int (\rho v_\theta^2)^* \left\{\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\left(e^{-ikR}\frac{1}{r_0}\frac{\partial I_R}{\partial\theta_0}\right)\right.$$
$$-ik\left(\frac{1}{r_0^2}\frac{\partial^2 R}{\partial\theta_0^2}+\frac{1}{r_0}\frac{\partial R}{\partial\theta_0}\right)e^{-ikR}-ik\frac{1}{r_0}\frac{\partial R}{\partial\theta_0}\frac{1}{r_0}\frac{\partial I_R}{\partial\theta_0}e^{-ikR}$$
$$\left.-k^2\left(\frac{1}{r_0}\frac{\partial R}{\partial\theta_0}\right)^2 e^{-ikR}I_R+\frac{1}{r_0}e^{-ikR}\frac{\partial I_R}{\partial r_0}\right\}\frac{dV_0}{R}. \quad (17)$$

Derivatives of $1/R$ have been omitted because they do not appear in the farfield. Neither does the term containing

$$\frac{1}{r_0^2}\frac{\partial^2 R}{\partial\theta_0^2}+\frac{1}{r_0}\frac{\partial R}{\partial\theta_0}$$

for it is of order $1/R^2$, as can be verified by performing the differentiations. Now

$$e^{-ikR}\frac{1}{r_0}\frac{\partial I_R}{\partial\theta_0} = e^{-ikR}\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}e^{-iu_R^2}\frac{1}{r_0}\frac{\partial u_R}{\partial\theta_0} = k\frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}}e^{-ikD}(kr_0)^{-\frac{1}{2}}\sin\tfrac{1}{2}(\theta-\theta_0)\,(\sin\phi)^{\frac{1}{2}},$$

so that $$\frac{1}{r_0}\frac{\partial}{\partial\theta_0}e^{-ikR}\frac{1}{r_0}\frac{\partial I_R}{\partial\theta_0} = -\tfrac{1}{2}k^2\frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}}e^{-ikD}(kr_0)^{-\frac{3}{2}}\cos\tfrac{1}{2}(\theta-\theta_0)\,(\sin\phi)^{\frac{1}{2}}.$$

Also, $$e^{-ikR}\frac{1}{r_0}\frac{\partial I_R}{\partial r_0} = k^2\frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}}e^{-ikD}(kr_0)^{-\frac{3}{2}}\cos\tfrac{1}{2}(\theta-\theta_0)\,(\sin\phi)^{\frac{1}{2}}.$$

Combining these results, we have

$$\int (\rho v_\theta^2)^* \left\{\frac{1}{r_0^2}\frac{\partial^2}{\partial\theta_0^2}+\frac{1}{r_0}\frac{\partial}{\partial r_0}\right\}\frac{e^{-ikR}}{R}I_R\,dV_0 = -k^2\int (\rho v_\theta^2)^* \left\{\left(\frac{1}{r_0}\frac{\partial R}{\partial\theta_0}\right)^2 e^{-ikR}I_R\right.$$
$$-i\frac{1}{r_0}\frac{\partial R}{\partial\theta_0}\frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}}e^{-ikD}(kr_0)^{-\frac{1}{2}}\sin\tfrac{1}{2}(\theta-\theta_0)\,(\sin\phi)^{\frac{1}{2}}$$
$$\left.+\frac{1}{2}\frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}}e^{-ikD}(kr_0)^{-\frac{3}{2}}\cos\tfrac{1}{2}(\theta-\theta_0)\,(\sin\phi)^{\frac{1}{2}}\right\}\frac{dV_0}{R}. \quad (18)$$

The remaining terms on the right-hand side of (7) can be dealt with similarly.

Figure 2 shows a sketch of the behaviour of $I_R(I_{R'})$ as a function of $u_R(u_{R'})$. It is apparent that if $(kr_0)^{\frac{1}{2}}$ is very large and $u_R(u_{R'})$ is not large and negative then the terms containing $I_R(I_{R'})$ are much larger than those containing $(kr_0)^{-\frac{1}{2}}$ and $(kr_0)^{-\frac{3}{2}}$. The signs of u_R and $u_{R'}$ depend upon the signs of $\cos\frac{1}{2}(\theta-\theta_0)$ and $\cos\frac{1}{2}(\theta+\theta_0)$ respectively. If both $\cos\frac{1}{2}(\theta-\theta_0)$ and $\cos\frac{1}{2}(\theta+\theta_0)$ are positive, and this will be the case if θ is acute and $\theta_0 \leqslant \pi-\theta$, then

$$4\pi p^* = -k^2\int (\rho v_i v_j)^*\left[R_i R_j\frac{e^{-ikR}}{R}I_R+R_i' R_j'\frac{e^{-ikR'}}{R'}I_{R'}\right]dV_0, \quad (19)$$

where $$(v_i) = (v_r, v_\theta, v_z),$$

$$(R_i) = \left(\frac{\partial R}{\partial r_0}, \frac{1}{r_0}\frac{\partial R}{\partial\theta_0}, \frac{\partial R}{\partial z_0}\right),$$

and $$(R_i') = \left(\frac{\partial R'}{\partial r_0}, \frac{1}{r_0}\frac{\partial R'}{\partial\theta_0}, \frac{\partial R'}{\partial z_0}\right).$$

If θ_0 is just a little greater than $\pi-\theta$ so that $u_{R'}$, although negative, still has $I_{R'}$ with magnitude much greater than $(kr_0)^{-\frac{1}{2}}$ then (19) is still the appropriate expression for p^*. For larger values of θ_0, but θ_0 still less than $\pi+\theta$, $I_{R'}$ has magnitude comparable to $(kr_0)^{-\frac{1}{2}}$ and so is negligible compared to I_R which is, at least, greater than $\frac{1}{2}$. When θ_0 is in this range the expression for p^* is

$$4\pi p^*(r,\theta,z;\omega) = -k^2\int(\rho v_i v_j)^* R_i R_j \frac{e^{-ikR}}{R} I_R\, dV_0. \tag{20}$$

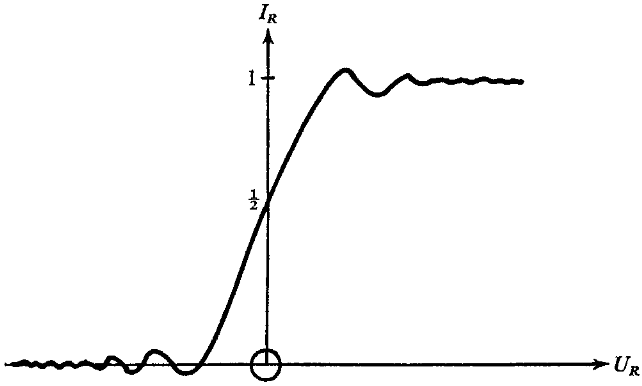

FIGURE 2. Sketch of the function I_R.

Again, if θ_0 is a little greater than $\pi+\theta_0$, I_R is still much greater than $(kr_0)^{-\frac{1}{2}}$ and (20) is still applicable. For larger θ_0, however, both I_R and $I_{R'}$ are comparable to $(kr_0)^{-\frac{1}{2}}$. The expression for p^* would now have to include not only the terms containing I_R and $I_{R'}$ but also those containing the factor $(kr_0)^{-\frac{1}{2}}$. We shall not write this expression down but merely note that the value of p^* which it would predict is much smaller, by the factor $(kr_0)^{-\frac{1}{2}}$, from the values given by (19) or (20).

Now (see Pearcey 1956),

$$\frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}}\int_{-\infty}^{X} e^{-iu^2}du \sim 1+\frac{e^{\frac{1}{4}i\pi}}{2\sqrt{\pi}}\frac{e^{-iX^2}}{X} \quad \text{as} \quad X\to\infty, \tag{21}$$

so that, except near $\theta_0 = \pi-\theta$, we can write (19) as,

$$4\pi p^*(r,\theta,z;\omega) = -k^2\int(\rho v_i v_j)^*\left[R_i R_j\frac{e^{-ikR}}{R}+R'_i R'_j\frac{e^{-ikR'}}{R'}\right]dV_0, \tag{22}$$

and (20) as

$$4\pi p^*(r,\theta,z,\omega) = -k^2\int(\rho v_i v_j)^* R_i R_j\frac{e^{-ikR}}{R}\,dV_0, \tag{23}$$

except near $\theta_0 = \pi+\theta$.

The position for eddies far from the edge is summarized schematically in figure 3. Where (22) holds, the half plane behaves like an infinite rigid plane with the edge having a negligible effect of order $(kr_0)^{-\frac{1}{2}}$. Where (23) holds, the sound is just that of an eddy in free turbulence, the edge again having a negligible effect of order $(kr_0)^{-\frac{1}{2}}$. Finally, an eddy in the geometrical shadow of the field point produces sound pressures lower than those from free turbulence by the factor $(kr_0)^{-\frac{1}{2}}$. Between each of these three regions there is a region where the sound pressures are intermediate between those of the neighbouring regions. There is no sharp discontinuity between, say, the regions from where reflected and direct sound is heard and that from where only direct sound is heard. However, the angular (θ_0) extent of these transition regions decreases with

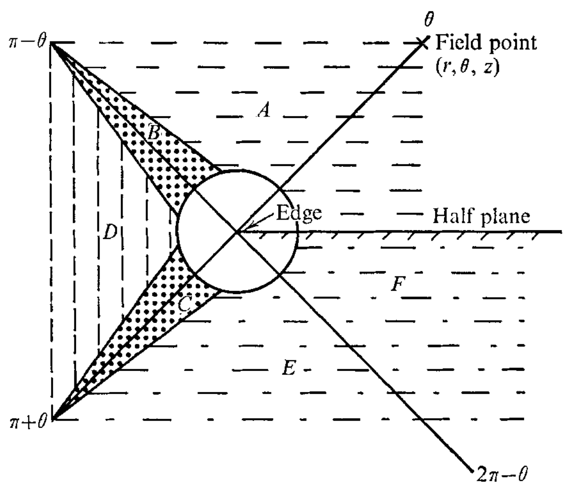

FIGURE 3. Sketch showing regions where the various equations are applicable for estimating the sound from eddies far from the edge of the half plane. Region: A, equation (22) holds if eddy is in this region; B, equation (19) holds; C, equation (20) holds; D, equation (23) holds; E, sources in this region produce sound pressures of p reduced by factor $(kr_0)^{\frac{1}{2}}$ on those from other regions; F, points further from the edge than this satisfy the inequality $(kr_0)^{\frac{1}{2}} \gg 1$.

increasing distance of the eddy from the edge. The narrower this transition region, the sharper does the contrast between the sound heard from one region become with that from another.

If an eddy is supported by the half plane and is on the same side of it as the field point, then there is at most a fourfold increase in the sound intensity at the field point: if the eddy is on the opposite side of the half plane, then the intensity is reduced by the large factor $(kr_0)^{-\frac{1}{2}}$ and may be taken as zero. Lastly, if the eddy is far from both the edge and the half plane (but not in the shadow region) then the intensity is the same as if the turbulence was unbounded. This is true even when there is reflected sound for then the travel time of the reflected sound is many eddy lifetimes longer than the travel time of the direct sound.

The essential point is that, except when the eddy lies in the geometrical shadow of the field point, a turbulent eddy far from the edge produces a sound of intensity comparable to that of an eddy in free turbulence.

5. The effect of motion of the half plane

Until now the half plane has been supposed perfectly rigid. However, it is possible to discuss the case of a half plane which is sufficiently limp that it cannot support any normal stress, that is, a pressure release boundary.

The Fourier transformed version of Lighthill's equation, our equation (3), now has to be solved subject to the condition that $p^* = 0$ on the half plane.

A formal solution of (3) is now

$$p^* = \int \left(\frac{\partial^2 v_i v_j}{\partial y_i \partial y_j}\right)^* \tilde{G}\, dV, \tag{24}$$

where $(\nabla^2 + k^2)\tilde{G} = -4\pi\delta(\mathbf{x}-\mathbf{y})$, and $\tilde{G} = 0$ on the half plane. Macdonald (1915) has shown that

$$\tilde{G} = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}\left\{\frac{e^{-ikR}}{R}\int_{-\infty}^{u_R} e^{-iu^2}\,du - \frac{e^{-ikR'}}{R'}\int_{-\infty}^{u_{R'}} e^{-iu^2}\,du\right\},$$

where the symbols have their previous significance. The only difference between G and $\tilde{G}$ is the change of sign of the 'image' part of G.

If the divergences are completed in (24) then

$$p^* = \int (\rho v_i v_j)^* \frac{\partial^2 \tilde{G}}{\partial y_i \partial y_j}\, dV - \int \rho v_n^2 \frac{\partial \tilde{G}}{\partial n}\, dS.$$

The surface integral is to be evaluated over the mean position of the half plane. We may omit it from further discussion by observing that because only one differentiation of $\tilde{G}$ is involved the integral will contain the factor $(2kr_0)^{-\frac{1}{2}}$ which means that when $2kr_0 \ll 1$ it is negligible compared to the terms containing $(2kr_0)^{-\frac{3}{2}}$ arising from the double differentiations of the volume integrals. On the other hand, when $(kr_0)^{\frac{1}{2}} \gg 1$ the surface integral is negligible compared to the terms retained in the volume integral. The subsequent analysis is almost identical to that given for a rigid half plane.

For turbulence very near the edge we find

$$4\pi p^* = k^2 \frac{2e^{\frac{1}{4}i\pi}}{\sqrt{\pi}} \sin\tfrac{1}{2}\theta\,(\sin\phi)^{\frac{1}{2}}$$
$$\times \int \{\rho v_r^2 \sin\tfrac{1}{2}\theta_0 - \rho v_\theta^2 \sin\tfrac{1}{2}\theta_0 + 2\rho v_r v_\theta \cos\tfrac{1}{2}\theta_0\}^* (2kr_0)^{-\frac{3}{2}} \frac{e^{-ikR}}{R}\, dV_0. \tag{25}$$

This should be compared with (10). The only significant difference is the replacement of the directional factor $\cos\frac{1}{2}\theta$ in (10) by $\sin\frac{1}{2}\theta$.

Again, when the turbulence is far from the edge, the only difference from the rigid case is that any reflected sound has a change of phase of π.

With both the rigid and the pressure release half planes having essentially the same effect on the turbulence produced noise, it seems reasonable to conjecture that homogeneous half planes of intermediate properties also do so.

6. General implications of the theory

We have seen that eddies close to the edge of a half plane are much more powerful sources of sound than eddies far from the edge. The intensity at a farfield point of the sound from a single eddy near the edge is given approximately by the formula (14) which, if the directional factors are suppressed, can be written

$$I = \frac{k^4 \rho_0 U^4 \alpha^2 \sin^2 \bar{\theta}\, V^2}{\pi^3 c R^2 (k\bar{r}_0)^3} \sim \rho U^3 \left(\frac{U}{c}\right)^2 \frac{\delta^2}{R^2}. \tag{26}$$

The corresponding formula for an eddy far from the edge (not in the shadow region) is equivalent to that of an eddy in free space

$$I = \frac{k^4 \rho_0 U^4 \alpha^2 V^2}{32\pi^2 c R^2} \sim \rho U^3 \left(\frac{U}{c}\right)^5 \frac{\delta^2}{R^2}. \tag{27}$$

We are now in a position to estimate the scales of a surface in the critical case when the surface sound has an intensity comparable to that arising from the eddies near the edge. Larger surfaces than this critical size will be essentially unaffected by the edges while smaller surfaces will be dominated by the edge noise.

Comparing (26) and (27) we see that an eddy near the edge radiates an acoustic intensity equivalent to that of $(32/\pi) \times (k\bar{r}_0)^{-3}$ eddies in free space or a quarter this number of eddies in the boundary layer far from the edge. A minimum estimate of $\bar{r}_0$ is the correlation radius δ of the eddy, corresponding to an eddy centred on the edge. An eddy further from the edge will give rise to an intensity appreciably lower than this closest eddy because of the dependence of the intensity on the third power of the distance from the edge. For instance, a doubling of the effective distance of an eddy from the edge results in a 9 db lowering of the intensity. Taking, however, δ as our estimate for $\bar{r}_0$ the edge region is equivalent, in sound generating ability, to the area within

$$(16/\pi)\,(k\delta)^{-3} \times 2\delta$$

of the edge. Since $k\delta$ is of order πM the surfaces must have dimension normal to the edge in excess of $(16/\pi^4)\,M^{-3}\delta$, or, equivalently, $(8/\pi^5)\,M^{-2}$ acoustic wavelengths if the edge effect is to be other than dominant. This is necessarily a minimum estimate, for no account has been taken of the eddies farther from the edge than the closest eddy but still well within a wavelength of the edge, but it does show that if the fluid is water where M is seldom above 10^{-2} then to all intents and purposes the flow noise must come from the edge. Real underwater surfaces are unlikely to be large enough for surface noise to be important.

A further consideration of some practical importance can be deduced from (26). The factor $\sin^2 \bar{\theta}$ reduces the scattered noise from flows which pass obliquely over the edge. It is difficult to justify the transfer of results obtained for such an ideal surface as an infinitely thin, rigid half plane to surfaces which are encountered in the real world, but this does suggest that the noise from a sharp-edged surface can be considerably reduced by giving it a swept wing characteristic. Marine propellers with greater curvature in the span-wise direction might be expected to generate less noise than those which have the span-wise direction more or less radial.

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REFERENCES

CURLE, N. 1955 The influence of solid boundaries upon aerodynamic sound. *Proc. Roy. Soc. Lond.* A **231**, 505–514.

LIGHTHILL, M. J. 1952 On sound generated aerodynamically. I. General theory. *Proc. Roy. Soc. Lond.* A **211**, 564–587.

LIGHTHILL, M. J. 1954 On sound generated aerodynamically. II. Turbulence as a source of sound. *Proc. Roy. Soc. Lond.* A, **222**, 1–32.

PEARCEY. T. 1956 *Table of the Fresnel Integral.* Cambridge University Press.

MACDONALD, H. M. 1915 A class of diffraction problems. *Proc. Lond. Math. Soc.* (2), **14**, 410–427.